Outline: Finding Parametric Equations

1. **Line Segments**

   The line segment from a point $\vec{a}$ to a point $\vec{b}$ can be parameterized as follows:

   $$\vec{x}(t) \;=\; \vec{a} + t(\vec{b} - \vec{a}).$$

   Note that $\vec{x}(0) = \vec{a}$ and $\vec{x}(1) = \vec{b}$. This formula can also be written

   $$\vec{x}(t) \;=\; (1 - t)\vec{a} + t\vec{b}.$$

2. **Circles and Ellipses**

   The unit circle is parameterized by $\vec{x}(t) = (\cos t, \sin t)$. More generally, a circle with radius r and center (a, b) is given by

   $$\vec{x}(t) \;=\; (a + r\cos t,\; b + r\sin t).$$

   Even more generally, an ellipse whose axes are horizontal and vertical is given by

   $$\vec{x}(t) \;=\; (a + r_x \cos t,\; b + r_y \sin t).$$

   where r_x is the horizontal radius, and r_y is the vertical radius.

   Sometimes you need to parameterize circular motion starting at the top, left, or right of a circle, or perhaps motion that goes clockwise. Such a parametrization can be obtained from the standard parametrization by (1) possibly switching the sine and the cosine, and (2) negating the sine, the cosine, both, or neither.

3. **Graphs of Functions**

   A curve of the form $y = f(x)$ can be parameterized by

   $$\vec{x}(t) \;=\; \big(t, f(t)\big).$$

   A polar curve of the form $r = f(\theta)$ can be parameterized by

   $$\vec{x}(t) \;=\; \big(f(t)\cos t, f(t)\sin t\big).$$

4. **General Strategy**

In general, when trying to parameterize a complicated curve, use the following strategy:

1. Instead of thinking about the motion, just draw a picture of the situation at time t, and see if you can figure out where everything is.
2. It often helps to compute unit vectors in the important directions, and then express other vectors as linear combinations of these unit vectors.

5. **Rolling**

The main principle of rolling is the following:

> **When two curves roll against each other, the distances rolled along each curve are always equal.**

This rule is usually required to analyze any rolling situation.

6. **Envelopes**

An **envelope** is a curve tangent to a family $L(t)$ of lines. Parametric equations for an envelope can be found using the formula

$$\vec{x}(t) = \lim_{h \to 0} \vec{P}(t,h)$$

where $\vec{P}(t,h)$ is the point where $L(t)$ and $L(t+h)$ intersect.

7. **Pedal Curves**

A **pedal curve** is a curve obtained by projecting a fixed point onto a moving line. To parametrize a pedal curve, just draw the situation at time t, and then use the dot product to figure out the projection.

8. **Differential Equations**

Some curves are determined by differential equations, or even systems of differential equations. Just remember:

1. When setting up a differential equation from geometry, you may need to use the fact that the slope $\dfrac{dy}{dx}$ of a parametric curve is the same as $\dfrac{y'(t)}{x'(t)}$.
2. Use separation of variables to solve first-order equations.
3. If a second-order equation involves y'' and y' but not y, make the substitution $u = y'$ to change it into a first-order equation.